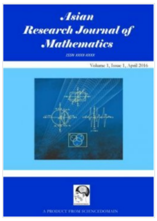

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Parameter Estimation of Exponentiated U-Quadratic Distribution: Alternative Maximum Likelihood and Percentile Methods

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this article, exponentiated U-quadratic distribution (EUq) is proposed by exponentiation procedure. The quantile function and r^{th} moment of the new model are computed. Estimation of parameter by the alternative maximum likelihood and estimation based on percentiles are established and compare their performances through numerical simulations. The results show that both methods are suitable for the parameter estimation of EUq distribution. A real data set is used to compare the fit of the two proposed estimation method using Kolmogorov Smirnov test (KS).

Keywords: U-qadrtic distribution; maximum likelihood estimation; estimation by percentile.

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1 Introduction

Estimation of parameters of probability distributions has been achieved through maximum likelihood, least square, percentile, Bayesian method among others. For example, recently [1] consider the estimation of parameters of Kumaraswamy (Kw) distribution by ten different approaches including maximum likelihood and percentile methods. [2] compare the estimations in the generalized exponential Poisson (GEP) by several methods including maximum-likelihood and percentile. Estimation of the parameters of Weibull (W) via percentile method was analyzed by [3]. [4] consider a different method of estimation parameters of generalized exponential (GE) distribution including the maximum likelihood and percentile methods. [5] discussed the estimation of weighted exponential (WE) using five methods of parameter estimation including the maximum likelihood method.

This work proposed exponentiated U-quadratic distribution and its basic properties. The estimation of shape parameter of the exponentiated U-quadratic (EUq) distribution using alternative maximum likelihood and percentile procedures are established, the methods shown to provide a good estimate, and improved as the sample size increased. At the end, we compare the two methods by fitting a real data set.

The cumulative distribution function G(x) of the U-quadratic distribution is given by

$$G(x) = \frac{\alpha}{3}((x-\beta)^3 + (\beta - a)^3), \qquad x \in [a, b] \tag{1.1}$$

where $a \in (-\infty, \infty)$, $b \in (a, \infty)$, $\alpha = \frac{12}{(b-a)^3}$ and $\beta = \frac{a+b}{2}$. The corresponding pdf of (1.1) is

$$g(x) = \alpha(x-\beta)^2. \tag{1.2}$$

Recently, [6] show that the U-quadratic distribution can be considered as a proxy for a transformed triangular distribution. Moreover, some new extentions of the U-quadratic distribution known as the transmuted U-quadratic distribution (TUQ) was proposed by [7] and the exponentiated generalized U-quadratic (EGUq) distribution by [8].

The rest of the paper is organized as follows. In section 2, we define the exponentiated U-quadratic (EUq) distribution and derive some of its important properties. In section 3, Estimation of Parameter by the alternative maximum likelihood and percentile methods are discussed. In section 4, simulation studies and real data illustration are provided. Conclusions in section 5.

2 Exponentiated U-quadratic Distribution

Here, we proposed the Exponentiated U-quadratic distribution by exponentiation procedure. Over the years, several distributions have been proposed using exponentiation technique, for example, the exponentiated gumbel (EGu), exponentiated weibull (EW), exponentiated gamma (EG), and exponenciated prechlet (EFr) distributions were proposed by [9]. Moreover, the generalize exponential (GE) by [10], exponentiated generalize linear exponential (EGLE) by [11], exponentiated log-logictic (ELL) by [12], exponenciated weibull (EW) by [13], generalized BurrXII-Poisson(GBXIIP) by [14], and generalized half-poisson (GHLP) by [15] etc.

The cumulative distribution function $F(x)$ and the probability density function $f(x)$ of exponentiated U-quadratic distribution (EUq) are obtained respectively by

$$F(x) = (\frac{\alpha}{3})^\theta \left((x-\beta)^3 + (\beta - a)^3\right)^\theta \tag{2.1}$$

and

$$f(x) = \theta\alpha^\theta 3^{1-\theta}(x-\beta)^2 \left((x-\beta)^3 + (\beta - a)^3\right)^{\theta-1} \tag{2.2}$$

The survival function $s(x)$ of exponentiated U-quadratic distribution is given by

$$s(x) = 1 - (\frac{\alpha}{3})^{\theta} \left((x-\beta)^3 + (\beta - a)^3\right)^{\theta}$$

and the hazard function $h(x)$ of exponentiated U-quadratic distribution is

$$h(x) = \frac{\theta\alpha^{\theta}3^{1-\theta}(x-\beta)^2\left((x-\beta)^3 + (\beta-a)^3\right)^{\theta-1}}{1 - (\frac{\alpha}{3})^{\theta}\left((x-\beta)^3 + (\beta-a)^3\right)^{\theta}}$$

Figure 1 and 2 show the plot of the density function and hazard rate function of the exponentiated U-quadratic distribution for some values of θ respectively. The quantile function of exponentiated U-quadratic distribution can be obtained by inverting (2.1) as

$$Q(p) = \sqrt[3]{\left(\frac{3}{\alpha}\right)p^{\frac{1}{\theta}} - (\beta - a)^3} + \beta, \qquad p \in (0,1). \tag{2.3}$$

The quantile in (2.3) can be used for random data generation that follow the EUq distribution.

Proposition 2.1. *Let $U \sim U(0,1)$, where $U(0,1)$ is a uniform distribution, then* $X = \left(\left(\frac{3}{\alpha}\right)u^{\frac{1}{\theta}} - (\beta-a)^3\right)^{1/3} + \beta$*, is a random variable that follow $EUq(a,b,\theta)$.*

The $r^{th}-$moment of the EUq is an important measure that can be used to study various features and characteristics of the EUq such as mean, variance, skewness, kurtosis, moment generating function etc. The $r^{th}-$moment of the EUq is computed as follows.

Proposition 2.2. *Let $X \sim EUq(a,\, b,\, \theta)$, then, for $r \in \mathbb{N}$, the $r^{th}-$moment of X is given by*

$$\mu_r = \sum_{i=0}^{\infty}\sum_{j=0}^{3i+2}\psi_{r,j}(a,b,\theta)\left(b^{r+j+1} - a^{r+j+1}\right), \tag{2.4}$$

for $|\frac{x-\beta}{\beta-a}| < 1$, where $\psi_{r,j}(a,b,\theta) = \binom{\theta-1}{i}\binom{3i+2}{j}\frac{\theta\alpha^{\theta}3^{1-\theta}(-\beta)^{3i-j+2}(\beta-a)^{3(\theta+i-1)}}{r+j+1}$.

Proof.

$$\mu_r = \int_a^b x^r f(x)dx = \theta\alpha^{\theta}3^{1-\theta}\int_a^b x^r(x-\beta)^2\left((x-\beta)^3 + (\beta-a)^3\right)^{\theta-1}dx,$$

for θ real and non integer we can applying the generalized binomial expansion in $\left((x-\beta)^3 + (\beta-a)^3\right)^{\theta-1} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(x-\beta)^{3i}(\beta-a)^{3(\theta+i-1)}$, for $|\frac{x-\beta}{\beta-a}| < 1$, therefore,

$$\mu_r = \theta\alpha^{\theta}3^{1-\theta}\sum_{i=0}^{\infty}\binom{\theta-1}{i}(\beta-a)^{3(\theta+i-1)}\int_a^b x^r(x-\beta)^{3i+2}dx,$$

by expanding $(x-\beta)^{3i+2} = \sum_{j=0}^{3i+2}\binom{3i+2}{j}x^j(-\beta)^{3i-j+2}$, thus,

$$\mu_r = \theta\alpha^{\theta}3^{1-\theta}\sum_{i=0}^{\infty}\sum_{j=0}^{3i+2}\binom{\theta-1}{i}\binom{3i+2}{j}(\beta-a)^{3(\theta+i-1)}(-\beta)^{3i-j+2}\int_a^b x^{r+j}dx,$$

hence,

$$\mu_r = \sum_{i=0}^{\infty}\sum_{j=0}^{3i+2}\psi_{r,j}(a,b,\theta)\left(b^{r+j+1} - a^{r+j+1}\right),$$

where $\psi_{r,j}(a,b,\theta) = \binom{\theta-1}{i}\binom{3i+2}{j}\frac{\theta\alpha^{\theta}3^{1-\theta}(-\beta)^{3i-j+2}(\beta-a)^{3(\theta+i-1)}}{r+j+1}$.

□

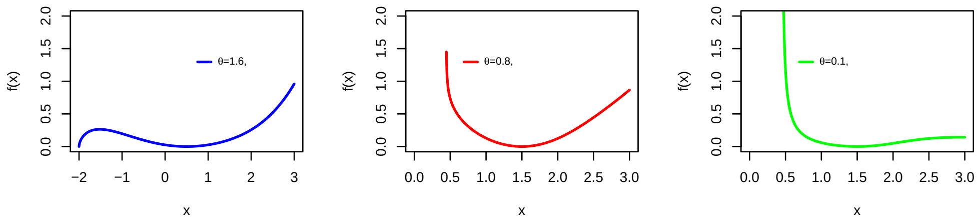

Fig. 1. Plot of the density function of the exponentiated U-quadratic distribution.

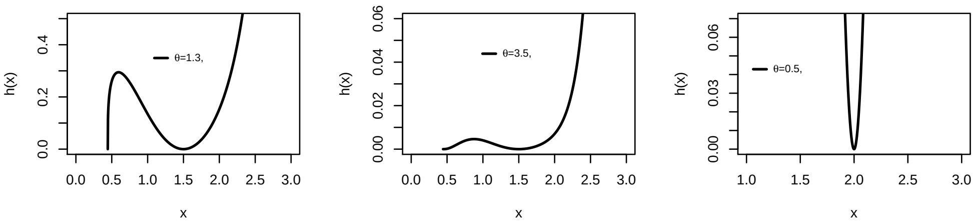

Fig. 2. Plot of the hazard rate function of the exponentiated U-quadratic distribution.

3 Methods of Estimation

In this section, we analyzed the two methods for estimating the parameter θ, of the exponentiated U-quadratic distribution. Throughout this work we assume that $x_1, x_2, x_3, \cdots, x_n$ is a random sample of size n obtaind from the exponentiated U-quadratic distribution with parameter θ unknown. We let $x_{1:n} \leq x_{2:n} \leq x_{3:n} \leq \cdots \leq x_{n:n}$ denote the associated order statistics from the sample.

3.1 Alternative maximum likelihood method

In this subsection, we consider the alternative maximum likelihood method (AMLE) due to the irregularity of the usual maximum likelihood estimation (MLE) which occurs at any point corresponding to the minimum order statistics. The MLE method has nice properties of being consistent, asymptotically efficient under very general conditions. However, in unbounded likelihood problem like estimation of some continuous distributions the MLE method does not always give satisfactory results see [16; 17; 18; 19]. In the alternative maximum likelihood method (AMLE) method, in this case, we set $a = x_1$, and excluded all the data points correspond to x_1 from the sample and use the usual maximum likelihood method. For a random sample $x_1, x_2, x_3, \cdots, x_n$ of size n the log-likelihood function of EUq can be written as :

$$\log\left(L(\theta)\right) = n\log\theta + n\theta\log\alpha + n(1-\theta)\log 3 + 2\sum_{i=1}^{n}\log(x_i - \beta) + (\theta - 1)\sum_{i=1}^{n}\log\left((x_i-\beta)^3 + (\beta - a)^3\right) \quad (3.1)$$

thus, we have

$$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{n}{\theta} + n\log\alpha - n\log 3 + \sum_{i=1}^{n} \log\left((x_i-\beta)^3 + (\beta-a)^3\right) \tag{3.2}$$

hence, the MLE of θ, say $\hat{\theta}_{MLE}$ is the solution of (3.2) as

$$\hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \log\left(\frac{\alpha}{3}\left((x_i-\beta)^3 + (\beta-a)^3\right)\right)}, \tag{3.3}$$

and we can determine the AMLE of θ, say $\hat{\theta}_{AMLE}$ directly from (3.3) by excluded all the data points correspond to x_1 as

$$\hat{\theta}_{AMLE} = \frac{-n}{\sum_{x_i \neq x_1}^{n} \log\left(\frac{\alpha}{3}\left((x_i-\beta)^3 + (\beta-a)^3\right)\right)}, \tag{3.4}$$

The following describe the irregularity of the usual MLE method, thus, the usual MLE fail to exist for EUq distribution.

Proposition 3.1. *Let $x_1, x_2, ..., x_n$ be an independent and identically random sample of size n form EUq, let $x_1 \leq x_2 \leq x_3 \leq ... \leq x_n$, be the order statistic obtain from the sample, then, there always exists a minimum order statistics for $x_i = x_j$ or $x_i \neq x_j$, and the log likelihood function in (3.1) diverges at x_1 i.e $\log L(\theta) \to -\infty|_{x_1}$ for $\theta > 1$ and $\log L(\theta) \to \infty|_{x_1}$ for $\theta < 1$.*

Proof. At $x_i = x_1 = a$, $(a-\beta)^3 + (\beta-a)^3 = 0$, thus, by considering the last term in (3.1), $\log L(\theta)$ diveges always and $\hat{\theta}_{MLE} \to 0$ for any random sample, hence the usual maximum likelihood method (MLE) fail to exist. □

Corollary 3.1. *If $\tau = \{f(x|\Theta) : \Theta \in \Omega\}$ is a family of distributions that contains the EUqs as a subfamily, then the maximum likelihood estimate of the parameter vector Θ based on an i.i.d. sample of size $n \geq 1$ drawn from $f(x|\Theta)$ does not exist.*

Proof. The fact that Ω contains the EUqs as a subfamily guarantees the existence of $\Omega^* \subset \Omega$ such that $\tau_0 = \{f(x|\Theta) : \Theta \in \Omega^*\}$ is the family of EUqs. Let $L(\Theta|x)$ denote the likelihood function for $f(x|\Theta)$, $\Theta \in \Omega$. The fact that the log likelihood function for the EUq is unbounded guarantees that $\log L(\Theta|x)$ is unbounded on Ω^*. □

3.2 Estimation based on percentile

The exponentiated Uquadratic distribution has an explicit cumulative distribution function, thus, the unknown parameter θ can be estimated by equating the sample percentile points with the population percentile points. For the random sample $x_1, x_2, x_3, \cdots, x_n$ of size n, let q_i be the estimate of $F(x;\theta)$, then, the percentile estimator of θ say $\hat{\theta}_p$ can be obtained from

$$q = \left[\left(\frac{\alpha}{3}\right)\left((p-\beta)^3 + (\beta-a)^3\right)\right]^{\theta}$$

thus,

$$\hat{\theta}_p = \frac{ln(q)}{ln\left[\left(\frac{\alpha}{3}\right)\left((p-\beta)^3 + (\beta-a)^3\right)\right]} \tag{3.5}$$

Remark 3.1. Notice that, for a random sample $(i.i.d)$ for which there exist p correspond to x_1, then the percentile method fail at p. thus, we avoid using such p.

4 Simulation Study

We assessed the proposed AMLE and percentile methods by simulation studies. 10,000 samples are generated of size $n = (10, 30, 60, 100, 150)$ form EUq for different values of a, b and θ. The estimated values, standard deviation (sd), bias and mean square error (MSE) of the estimators are computed using R-software. The bias and MSE are computed from

$$\mathbf{Bias}(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\theta}_i - \theta) \quad and \quad \mathbf{MSE}(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\theta}_i - \theta)^2.$$

In the case of percentile the median is used throughout the simulation i.e $p = 0.5$. The numerical values of the resulting simulation are given in Table 1. The resulting simulation shows that, in the two methods, (i) the standard deviation decreases as the sample size increases, (ii) the bias is decreasing as sample sizes increases, (iii) In both methods the MSE is decreasing as sample sizes increases, (iv) the AMLEs are positively bias or negative bias in some cases while (v) the percentile estimator is all positively biased in this case. Figs. 3 and 4 below show the plot of the bias and MSE of the resulting simulation respectively.

Table 1. AMLEs, percentile estimates, standard deviations, bias and MSE for various values of parameter

Sample size	Actual values			Estimated values		Standard deviations		Bias deviations		Mean square errors	
n	a	b	θ	$\hat{\theta}_p$	$\hat{\theta}_{amle}$	$sd(\theta_p)$	$sd(\theta_{amle})$	$Bias(\theta_p)$	$Bias(\theta_{amle})$	$MSE(\theta_p)$	$MSE(\theta_{amle})$
10	-3.0	5.0	0.5	0.5879	0.7940	0.2891	0.2998	0.0879	0.2940	0.0913	0.1763
	2.0	7.0	1.0	1.1257	1.4539	0.5472	0.6257	0.1257	0.4540	0.3152	0.5976
	-2.0	3.0	1.2	1.3408	1.7196	0.6625	0.7790	0.1408	0.5196	0.4587	0.8767
	-10	-5.0	1.8	2.0047	2.4428	1.0286	1.0982	0.2047	0.6428	1.0998	1.6191
	0.0	6.0	0.8	0.9137	1.2012	0.4435	0.5027	0.1137	0.4012	0.2096	0.4136
	4.0	20	3.0	3.3448	3.4365	1.6854	1.5419	0.3448	0.4365	2.9591	2.5677
30	-3.0	5.0	0.5	0.5258	0.5920	0.1456	0.1119	0.0258	0.0920	0.0219	0.0210
	2.0	7.0	1.0	1.0423	1.1123	0.2792	0.2212	0.0423	0.1123	0.0797	0.0615
	-2.0	3.0	1.2	1.2423	1.3026	0.3375	0.2696	0.0423	0.1026	0.1157	0.0832
	-10	-5.0	1.8	1.8791	1.8510	0.5131	0.4103	0.0791	0.0510	0.2694	0.1709
	0.0	6.0	0.8	0.8403	0.9091	0.2263	0.1773	0.0403	0.1091	0.0529	0.0433
	4.0	20	3.0	3.1293	2.9304	0.8392	0.7090	0.1293	-0.0696	0.7209	0.5075
60	-3.0	5.0	0.5	0.5129	0.5479	0.0967	0.0715	0.0129	0.0479	0.0095	0.0074
	2.0	7.0	1.0	1.0186	1.0536	0.1935	0.1423	0.0186	0.0536	0.0378	0.0231
	-2.0	3.0	1.2	1.2254	1.2389	0.2298	0.1692	0.0254	0.0389	0.0535	0.0302
	-10	-5.0	1.8	1.8425	1.7610	0.3467	0.2660	0.0425	-0.0390	0.1220	0.0723
	0.0	6.0	0.8	0.8193	0.8580	0.1530	0.1139	0.0193	0.0580	0.0238	0.0163
	4.0	20	3.0	3.0554	2.7882	0.5698	0.4652	0.0554	-0.2118	0.3277	0.2612
100	-3.0	5.0	0.5	0.5066	0.5314	0.0744	0.0539	0.0067	0.0314	0.0056	0.0039
	2.0	7.0	1.0	1.0151	1.0318	0.1460	0.1065	0.0151	0.0318	0.0216	0.0123
	-2.0	3.0	1.2	1.2148	1.2196	0.1763	0.1271	0.0148	0.0196	0.0313	0.0165
	-10	-5.0	1.8	1.8259	1.7478	0.2637	0.2012	0.0259	-0.0522	0.0702	0.0432
	0.0	6.0	0.8	0.8119	0.8368	0.1180	0.0855	0.0119	0.0368	0.0141	0.0087
	4.0	20	3.0	3.0472	2.7458	0.4414	0.3559	0.0472	-0.2542	0.1970	0.1913
150	-3.0	5.0	0.5	0.5043	0.5213	0.0595	0.0425	0.0043	0.0213	0.0036	0.0023
	2.0	7.0	1.0	1.0095	1.0289	0.1208	0.1046	0.0095	0.0289	0.0147	0.0118
	-2.0	3.0	1.2	1.2090	1.2083	0.1418	0.1011	0.0090	0.0083	0.0202	0.0103
	-10	-5.0	1.8	1.8161	1.7457	0.2148	0.1999	0.0161	-0.0544	0.0464	0.0167
	0.0	6.0	0.8	0.8085	0.8362	0.0951	0.0846	0.0085	0.0362	0.0091	0.0085
	4.0	20	3.0	3.0238	2.7456	0.3581	0.2842	0.0238	-0.2544	0.1288	0.1455

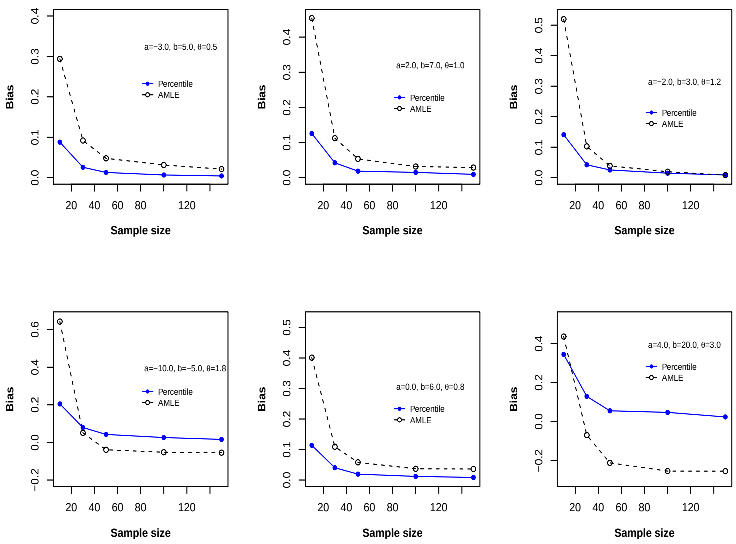

Fig. 3. Plots of the Bias given in Table 1

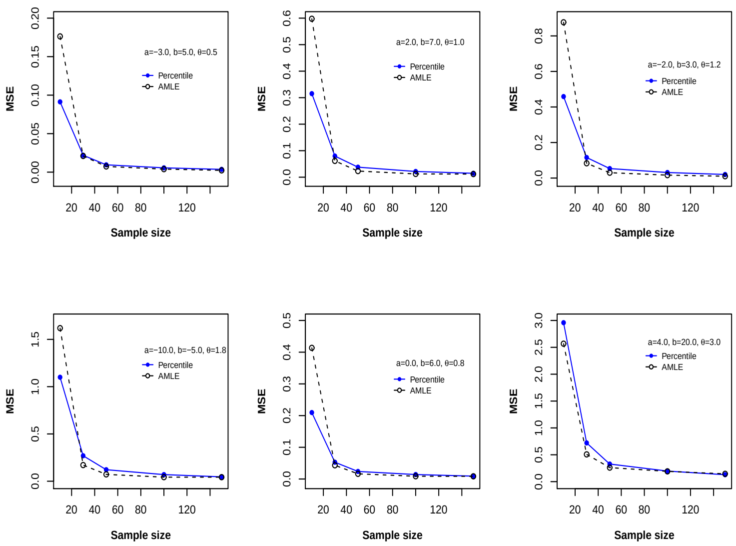

Fig. 4. Plots of the MSE given in Table 1

4.1 Illustration

In this subsection, the goodness of fit test statistics known as Kolmogorov Smirnov (KS) statistics is used to compare the fit of the EUq through the two estimation methods. The data set is the lifetimes of fifty 50 devices provided by [20] and recently studied by [21] : 0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Table 2. Estimators and the numerical value of the KS and p-value.

	θ	a	b	KS	p-value
EUq_p	1.0030	0.1	86	0.1382	0.2693
EUq_{AMLE}	0.9692	0.1	86	0.1501	0.1897

From Table 2 both the method are good for the estimation of the EUq, though, the percentile method has the smallest value of the KS, thus percentile perform better in this case. Figure 5 shows the plots of the empirical and estimated cumulative distribution while figure 6 display the quantile-quantile plot of the EUq estimated by (i) AMLE method and (ii) using percentile method.

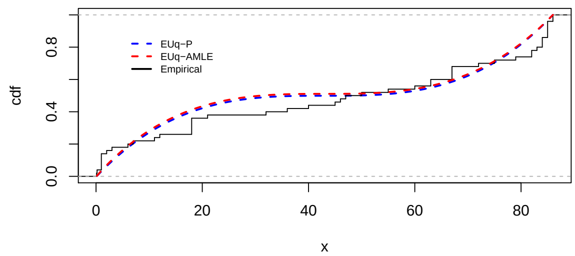

Fig. 5. Plots of the emperical and estimated cumulative distribution (cdf) for the given data set fitted using AMLE and percentile method.

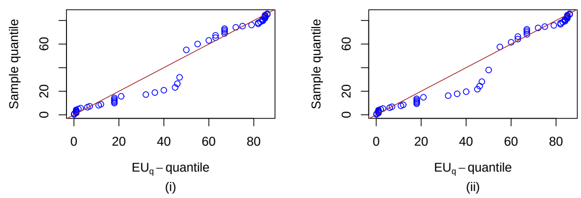

Fig. 6. Quantile quantile plots for the given data set (i) using AMLE method (ii) using percentile method.

5 Conclusion

We have proposed exponentiated U-quadratic distribution. The density function, survival function, hazard function, quantile function and r^{th} moment of the new model are presented. Estimation of the parameter by the alternative maximum likelihood and estimation based on percentiles are established and compare their performances through numerical simulations studies. According to the simulation, their performance improved as the sample size increased. A real data set is used to compare the fit of the two proposed estimation method using Kolmogorov Smirnov test (KS). The result shows that both methods are suitable for the parameter estimation of EUq distribution. But percentile method fit the data better with the slight difference in the KS value.

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Competing Interests

Authors have declared that no competing interests exist.

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